Ground-state and elemental excitations of the one-dimensional multicomponent Fermi gas with delta -function interaction

1994 J. Phys.: Condens. Matter 6 1359

(http://iopscience.iop.org/0953-8984/6/7/008)

J. Phys.: Condens. Matter 6 (1994) 1359–1378. Printed in the UK

Ground-state and elemental excitations of the one-dimensional multicomponent Fermi gas with δ-function interaction

P Schlottmann

Department of Physics, Florida State University, Tallahassee, FL 32306, USA

Received 25 October 1993

Abstract. We consider a gas of fermions with parabolic dispersion and N spin components (or spin S, $N = 2S + 1$) with SU(N) symmetry in one dimension interacting via a δ-function potential. The model is integrable and its solution has been obtained by Sutherland in terms of N nested Bethe *ansatze*. The ground-state Bethe *ansatz* integral equations are solved numerically for both repulsive and attractive interactions to obtain the energy, the chemical potential, and the magnetic susceptibility as a function of the band filling and the interaction strength. For the repulsive interaction the Fermi gas has the properties of a Luttinger liquid. In the attractive case, on the other hand, the fermions in the ground-state form bound states of up to N fermions of different spin components. The spectrum of elemental charge and spin excitations is derived for the repulsive and attractive situations. The spectrum is discussed in the limits of vanishing interaction strength and very strong coupling. For the repulsive interaction the low-lying charge excitations can be characterized by the Fermi momentum and the Fermi velocity. The range of the spin-wave excitations is correlated with the Fermi momentum of the charges. The spin-wave velocity is inversely proportional to the magnetic susceptibility. The spin-wave excitations become soft in the infinite-repulsive-coupling limit. In the attractive case in zero field all excitation branches except that of bound states of N fermions have an energy gap. It requires a finite energy to break these bound states and hence there is no response to a field smaller than a critical field. The low-T specific heat is proportional to the temperature.

1. Introduction

We consider the gas of fermions with parabolic dispersion in one dimension with N spin components (colours with SU(N) symmetry) interacting via a δ-function potential of interaction strength c. The Hamiltonian is the following:

$$H = -\sum_{i=1}^{N_e} \left(\frac{\partial}{\partial x_i} \right)^2 + 2c \sum_{i<j} \delta(x_i - x_j) \tag{1.1}$$

where N_e is the number of particles, the mass of the particles is equated to $\frac{1}{2}$ and the sum in the interaction term is restricted to $i < j$ to avoid double counting. The Hamiltonian is independent of the colours of the particles, which are incorporated via the symmetry of the wavefunction. The model has been exactly diagonalized by means of Bethe's *ansatz* for the following situations. (i) If $N = N_e$ the system (for symmetrized wavefunctions) is the gas of bosons solved by Lieb and Liniger [1]. (ii) For $S = \frac{1}{2}$ the model was diagonalized by two nested Bethe *ansatze* by Gaudin [2] and Yang [3]. (iii) This result was extended by Sutherland [4] to an arbitrary number N of spin components (see also [5]).

0953-8984/94/071359+20$07.50

In this paper we present a detailed study of the ground-state properties and derive the spectrum of elemental excitations of the model for $c > 0$ and $c < 0$ for arbitrary N by numerically solving the Bethe *ansatz* equations. The ground-state Bethe *ansatz* integral equations for $c > 0$ were stated by Sutherland [4] and those for an attractive interaction by Takahashi [6]. The thermodynamic Bethe *ansatz* equations of the N-component gas for both signs of the interaction were derived and discussed in [7]. For a repulsive interaction we obtain that the ground-state properties and the excitations are those expected for a Luttinger liquid, while if the potential is attractive the fermions form bound states of up to N fermions of different colour with spin gaps in the excitation spectrum.

One motivation for this study is the fact that the one-dimensional t–J model, which is integrable only at the supersymmetric point ($t = J$) [8,9], behaves like a Luttinger liquid if $t > J$ while it has a spin gap for $t < J$ [10]. In model (1.1) a small value of c could parametrize the deviation from supersymmetry, $(t - J)$, in the low-electron-density limit of the t–J model ($c > 0$ corresponds to a Luttinger liquid, while $c < 0$ has spin gaps in the excitation spectrum; the continuum approximation is adequate for a low-electron-density lattice model). This may contribute to a deeper understanding of the nature of the phases and of the phase boundary in the one-dimensional t–J model, and hence of strongly correlated electron systems.

The rest of this paper is organized as follows. In section 2 we restate the ground-state Bethe *ansatz* equations for model (1.1) for a *repulsive* interaction in terms of the energy potentials for each class of states [4,7]. In section 3 we numerically solve the ground-state integral equations for $c > 0$ and calculate the energy, the chemical potential and the magnetic susceptibility as a function of the band filling and the coupling strength c. The limits $c \to 0$ and $c \to \infty$ are discussed. Some results for the low-field susceptibility and low-T specific heat are also reviewed. The dispersions of the elemental charge and spin excitations for repulsive coupling are obtained in section 4. In section 5 we consider the ground-state Bethe *ansatz* equations for an *attractive* interaction [6,7]. These equations are then numerically solved in section 6 to obtain the energy, the chemical potential, and critical fields as a function of band filling and c. The critical fields are the threshold fields to break up a bound state of N fermions with different colours. The low-T specific heat and the limit $c \to 0$ are discussed. The spectrum of elemental excitations for attractive c is discussed in section 7. Finally, concluding remarks follow in section 8.

2. Ground state Bethe *ansatz* equations: repulsive interaction

Sutherland's [4] solution of model (1.1) for N colour components consists of an iterative application of the Bethe–Yang hypothesis (generalized Bethe *ansatz*) [2,3], such that one colour is eliminated at each step, leading to N nested Bethe *ansatze*. Each Bethe *ansatz* gives rise to a set of rapidities, $\{\xi_\alpha^{(l)}\}$, $l = 0, \ldots, N-1$, with α being the running index within each set. $k_\alpha = \xi_\alpha^{(0)}$ are known as charge rapidities, and the other $N-1$ sets refer to the spin degrees of freedom. All rapidities within a given set have to be different to guarantee linearly independent wavefunctions, giving rise to Fermi statistics for the rapidities, i.e. to 'particles' and 'holes'. In the ground state and for a repulsive potential ($c > 0$) all rapidities are real.

In the thermodynamic limit the rapidities satisfy integral equations, which can be formulated in terms of energy potentials for each class of states, $\epsilon(k)$ for the charges and $\varphi^{(l)}(\xi)$, $l = 1, \ldots, N-1$, for the spin degrees of freedom. These potentials are the energies entering the Fermi distribution function for each class of rapidities. Hence, a negative

potential defines the particles of the band and a positive potential the holes. For a Zeeman splitting (no crystalline fields or other external potentials) the Bethe *ansatz* equations take the form [7]

$$\epsilon(k) = k^2 - \mu - \sum_{l=1}^{2S} \int_{|\xi|>B_l} \mathrm{d}\xi \; F_{N-l}(\xi - k) \; \varphi^{(l)}(\xi) + \int_{-Q}^{Q} \mathrm{d}k' \; \epsilon(k') \; G_S(k - k')$$

$$\varphi^{(l)}(\xi) + \sum_{q=1}^{2S} \int_{|\xi'|>B_q} \mathrm{d}\xi' \; K_{l,q}(\xi - \xi')\varphi^{(q)}(\xi') = l(N-l)H + \int_{-Q}^{Q} \mathrm{d}k \; F_{N-l}(\xi - k) \; \epsilon(k) \tag{2.1}$$

where μ is the chemical potential and H the magnetic field. Here $K_{l,q}(\xi) = G^{\exp}_{l,q}(\xi) - \delta_{l,q} \; \delta(\xi)$ and $G^{\exp}_{l,q}(\xi)$, $G_S(\xi)$ and $F_l(\xi)$ are the Fourier transforms of

$$\hat{G}^{\exp}_{l,q}(\omega) = \exp(|\omega c|/2) \; \frac{\sinh\big[(N - \max(l,q))\omega c/2\big] \; \sinh\big[\min(l,q)\omega c/2\big]}{\sinh(N\omega c/2) \; \sinh(\omega c/2)}$$

$$\hat{G}_S(\omega) = \exp(-|\omega c|/2) \; \frac{\sinh(S\omega c)}{\sinh(N\omega c/2)} \qquad \hat{F}_l(\omega) = \frac{\sinh(l\omega c/2)}{\sinh(N\omega c/2)}. \tag{2.2}$$

The kernel of the integral equations for spin degrees of freedom, $\hat{G}^{\exp}_{l,q}(\omega)$, has the characteristic form dictated by the SU(N) invariance and has appeared previously in related problems: (i) the SU(N)-invariant Heisenberg chain [11], (ii) the Coqblin–Schrieffer model [12], (iii) the degenerate Anderson model with excluded multiple occupancy of the f level [13], and (iv) the SU(N) generalization of the Hubbard model [14, 15] and the degenerate supersymmetric $t - J$ model [9, 11].

The integration limits Q and B_l are determined by the zeros of the potentials, i.e. $\epsilon(\pm Q) = 0$ and $\varphi^{(l)}(\pm B_l) = 0$ and are related to μ and H. The potential $\epsilon(k)$ is negative for $|k| < Q$ (particles) and positive elsewhere (holes), so that Q defines the Fermi surface of the charges. Similarly $\varphi^{(l)}(\xi) < 0$ for $|\xi| < B_l$ (particles) and positive elsewhere (holes). In zero field we then have $B_l = \infty$ for all l. The distribution density for each set of rapidities can be obtained from the energy potentials by differentiating with respect to μ,

$$\rho_h(k) + \rho(k) = -(1/2\pi) \; (\partial\epsilon/\partial\mu) \qquad \sigma^{(l)}_h(\xi) + \sigma^{(l)}(\xi) = -(1/2\pi) \; (\partial\varphi^{(l)}/\partial\mu) \tag{2.3}$$

where $\rho(k)$ refers to particles and $\rho_h(k)$ to holes in the charge band and similarly for spin degrees of freedom. The integral equations satisfied by the distribution densities are obtained by differentiating (2.1) with respect to μ. These integral equations do not explicitly depend on μ and H, other than through Q and B_l.

The ground-state energy is given by [4] (L is the length of the box)

$$E/L = \int_{-Q}^{Q} \mathrm{d}k \; k^2 \; \rho(k) = \int_{-Q}^{Q} \frac{\mathrm{d}k}{2\pi} \; \epsilon(k) + \mu N_e/L \tag{2.4a}$$

and the number n_l/L of electrons of each colour is determined via [4]

$$n = N_e/L = \int_{-Q}^{Q} \mathrm{d}k \; \rho(k) \qquad n_1 = n - \int_{-B_1}^{B_1} \mathrm{d}\xi \; \sigma^{(1)}(\xi)$$

$$n_l = \int_{-B_{l-1}}^{B_{l-1}} \mathrm{d}\xi \; \sigma^{(l-1)}(\xi) - \int_{-B_l}^{B_l} \mathrm{d}\xi \; \sigma^{(l)}(\xi). \tag{2.4b}$$

3. Ground-state properties: repulsive interaction

In zero field all spin projections have the same number of particles, i.e. $B_l = \infty$ for all l and consequently $\sigma_{1h}^{(l)} \equiv 0$. Hence, the first equation of (2.1) reduces to a single linear integral equation of the Fredholm type for $\epsilon(k)$ (or $\rho(k)$). The integration limit Q parametrizes the total number of fermions as a function of the chemical potential. Hence, by solving the integral equation for a given Q we obtain μ from the condition $\epsilon(Q) = 0$ and the ground-state energy and the electron density N_e/L via (2.4).

An analytic solution of the integral equation for $\rho(k)$ can be obtained in the limits $c \to 0$ and $c \to \infty$. For $c \to 0$ we have $\rho(k) = N/(2\pi)$ if $|k| \leqslant Q$, $\rho_h(k) = 1/(2\pi)$ if $|k| \geqslant Q$, and zero elsewhere, so that

$$E/L = (N/3\pi)Q^3 \qquad N_e/L = (N/\pi)Q \qquad \mu = Q^2. \tag{3.1a}$$

This limit corresponds to a free electron gas with N colour components. For $c \to \infty$, on the other hand, two particles cannot be at the same place at the same time (hard-core potential), so that $\rho(k) = 1/(2\pi)$ for $|k| \leqslant Q$, $\rho_h(k) = 1/(2\pi)$ for $|k| \geqslant Q$, and zero elsewhere. It follows that

$$E/L = (1/3\pi)Q^3 \qquad N_e/L = (1/\pi)Q \qquad \mu = Q^2. \tag{3.1b}$$

As expected this result is independent of the number of spin components N; hence, the result would be identical for bosons and fermions. In the $c \to \infty$ limit the charges (not the electron gas) behave as non-interacting fermions.

The solution of the integral equation obeyed by $\rho(k)$ for general c and Q can only be obtained numerically. This is conveniently performed by iteration, discretizing the integral (using about 100 points). Note that $\rho(k)$ and $\rho_h(k)$ are symmetric and non-negative functions of k. The kernel of the integral equation can be expressed in terms of digamma functions

$$G_S(k) = \frac{1}{Nc\pi}\mathrm{Re}\left[\Psi\left(1 + \mathrm{i}\frac{k}{Nc}\right) - \Psi\left(\frac{1}{N} + \mathrm{i}\frac{k}{Nc}\right)\right]. \tag{3.2}$$

Note that if we scale k with c the integral equation for ρ for a given N depends on only one parameter, namely Q/c. In figure 1(a) we show $\rho(k) + \rho_h(k)$ as a function of k for $Q = 1$, $N = 6$ and four different values of c. For $c \neq 0$ the density smoothly interpolates between charge particle and hole states. For $c = 0$, on the other hand, the density function has a step singularity and is non-analytical. This is consistent with Takahashi's conclusion [16] that for $S = \frac{1}{2}$ it is possible to analytically continue from attractive to repulsive c unless $M = N_e/2$. In figure 1(b) we display the particle density as a function of Q for $N = 6$ and the same values of c, and in figure 1(c) we show the energy density as a function of N_e/L. Note that for finite c the slope of N_e/L with Q is the same as for $c = 0$ for large Q (if $Q \gg c$ the interaction strength does not play an important role), while for small Q it has the same slope as for $c = \infty$.

The chemical potential is given by

$$\mu = \frac{\partial E}{\partial N_e} = Q^2 + \int_{-Q}^{Q} \mathrm{d}k\; G_S(Q - k)\; \epsilon(k). \tag{3.3}$$

This is the energy required to add or remove one electron from the system; a change in N_e/L implies a modified integration limit Q and hence a changed $\rho(k)$, so that alternatively

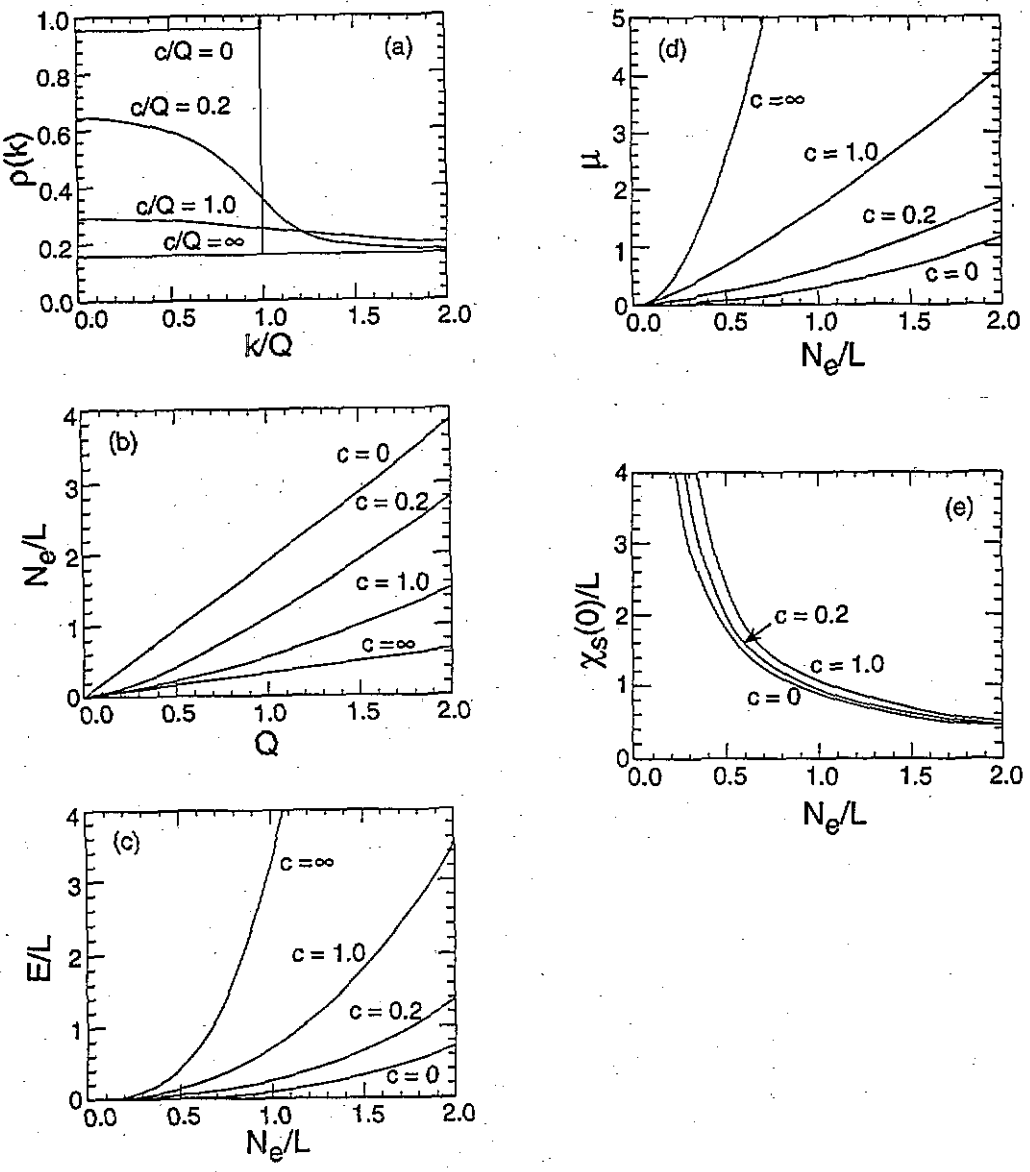

Figure 1. (a) Density distribution function $\rho(k)$ of the charge rapidities for the $N = 6$ Fermi gas with repulsive δ-function interaction. Q determines the Fermi surface for the charges. A step function at $k = \pm Q$ is obtained for non-interacting fermions. For $c \to \infty$ (hard-core potential) the distribution is constant. (b) Electron density N_e/L as a function of Q for $N = 6$ and four different couplings. (c) Energy density, (d) chemical potential and (e) zero-field spin susceptibility for $N = 6$ and $Q = 1$ as a function of the particle density for the same interaction strengths as before. The zero-temperature susceptibility diverges as $c \to \infty$ for all N_e/L. χ_S also diverges for all c as the electron density tends to zero as a consequence of the $k = 0$ van Hove singularity.

$\mu = (\partial E/\partial Q)/(\partial N_e/\partial Q)$. To obtain these partial derivatives an integral equation obeyed by $d\rho(k)/dQ$ (or by $d\epsilon(k)/dQ$), has to be solved numerically [15]. The chemical potential as a function of N_e/L is shown in figure 1(d).

The integral equation for $\rho(k)$ can be interpreted [17] as a new problem of particles without spin degrees of freedom interacting in one dimension with an effective potential

$$V(r) = -\frac{N-1}{2}c^2 \left(\sinh(Ncr/2)\right)^{-2} \tag{3.4}$$

where r is the relative coordinate between particles. This new many-body problem is integrable by construction and has been solved by Sutherland [18] using the results of Calogero and co-workers [19]. This problem is also closely related to the one-dimensional Toda lattice [20].

The zero-field susceptibility can be obtained in linear response to an arbitrarily small magnetic field [21, 22]

$$\chi_S(0)/L = -\frac{(N^2-1)N}{12}\,\frac{c}{\pi}\,\frac{\int_{-Q}^{Q} dk\ \rho(k)\ \exp(2\pi k/Nc)}{\int_{-Q}^{Q} dk\ \epsilon(k)\ \exp(2\pi k/Nc)}. \tag{3.5}$$

The susceptibility as a function of N_e/L for $N = 6$ and three values of c is shown in figure 1(e). As $N_e/L \to 0$ the susceptibility diverges as a consequence of the van Hove singularity of the empty band. For $c \to \infty$ the susceptibility diverges proportional to c/Q^2. This result is the consequence of (3.1*b*) being independent of N, i.e. the spin-wave excitations do not have dispersion and consequently χ_S tends to a Curie law. In the limit $c \to 0$, on the other hand, we obtain $\chi_S(0)/L = (N^2-1)/(24\pi Q) = \big((N^2-1)NL\big)/\big(24\pi^2 N_e\big)$. This expression explicitly shows the divergence as N_e/L tends to zero according to the $k = 0$ van Hove singularity.

If the magnetic field is small but finite logarithmic singularities arise in the zero-temperature susceptibility [21, 22]

$$\frac{\chi_S(H)}{\chi_S(0)} = 1 - \frac{1}{(N \ln H)} - \frac{\ln|N \ln H|}{(N \ln H)^2} + \cdots \tag{3.6}$$

this is a common feature of numerous models, e.g. the SU(N) Heisenberg chain [11, 22], the Babujian–Takhtajan model [23], the N-component supersymmetric $t-J$ model in one dimension [8, 9, 11], the Gross–Neveu model [24], and the SU(N) generalization of the Hubbard chain [14, 15, 17]. Any field lifting the degeneracy gives rise to the same form of logarithmic corrections, which are also a familiar feature in high-field expansions for the Kondo problem and the low-field dependence of undercompensated Kondo spins [12].

In the absence of symmetry-breaking fields, the energy bands for the spin rapidities are completely filled and there is no 'Fermi surface' for the spin-waves. An external degeneracy-lifting field introduces a 'Fermi surface' (a pair of points) in at least one of the spin-rapidity bands, even if the field is arbitrarily small. The logarithmic corrections are associated with an interference between the two Fermi points. Also the entropy behaves anomalously as both H and T tend to zero, and the γ coefficient of the low-temperature specific heat is different for states with and without 'Fermi surface', so the $H \to 0$ and $T \to 0$ limits cannot be interchanged [25].

4. Elemental charge and spin excitations: repulsive interaction

Charge excitations are obtained by adding (particles) or removing (holes) a rapidity from the set of charge rapidities $\{k_j\}$, while spin excitations (spin waves) correspond to adding or removing a spin rapidity $\xi^{(l)}$. On one hand, the excitation energies are given by the energy potentials $\epsilon(k)$ and $\varphi^{(l)}(\xi)$; on the other hand, they can be calculated (the traditional but tedious way) by computing the change in $\rho(k)$ when a rapidity is added or removed. Both methods yield identical results. Since the integral equations determining the density functions are linear, the linear superposition principle applies for any finite number of excitations. Their behaviour is soliton like: they are additive and independent. Excitation spectra for condensed matter related models have been previously derived for the Heisenberg chain [26,27], the traditional Hubbard model [28,29], the SU(N) invariant Hubbard model [15,30] and the supersymmetric $t-J$ model [8,9,31].

We consider the degenerate situation (no magnetic or crystalline fields), so that $B_l = \infty$ for $l = 1, \ldots, 2S$, i.e. all spin rapidity bands are full. If k_0 is the rapidity of the charge under consideration, we have a particle excitation if $|k_0| > Q$ and a hole excitation if $|k_0| < Q$, i.e.

$$\Delta E_{\mathrm{ch}}(k_0) = |\epsilon(k_0)| \tag{4.1}$$

with $k_0 = \pm Q$ representing the Fermi surface.

The integral equation for $\epsilon(k)$ can be solved analytically in the limiting cases $c \to 0$ and $c \to \infty$. For $c \to 0$ we obtain for the charge excitations

$$\Delta E_{\mathrm{ch}}(k_0) = \begin{cases} N\ (Q^2 - k_0^2) & \text{if} \quad |k_0| \leqslant Q \\ k_0^2 - Q^2 & \text{if} \quad |k_0| \geqslant Q \end{cases} \tag{4.2a}$$

while for $c \to \infty$ we have for all k_0

$$\Delta E_{\mathrm{ch}}(k_0) = |k_0^2 - Q^2|. \tag{4.2b}$$

Although the charge rapidities are frequently called momenta, they do not represent the physical momentum of the particles and holes, which is given by [9, 15, 28]

$$p = 2\pi \int_0^{k_0} \mathrm{d}k\ \left[\rho(k) + \rho_{\mathrm{h}}(k)\right]. \tag{4.3}$$

A charge removed with $k_0 = 0$ has zero momentum, $p(k_0 = 0) = 0$ and the momentum p is a monotonically increasing function of k_0 with the symmetry $p(-k_0) = -p(k_0)$. For $k_0 = Q$, (4.3) yields $p_{\mathrm{F}} = \pi N_{\mathrm{e}}/L$, the Fermi momentum. In the limit $c \to 0$ we obtain $p = Nk_0$ for $|k_0| < Q$ and $p = \pi N_{\mathrm{e}}/L + (k_0 - Q)$ for $k_0 > Q$. On the other hand, for $c \to \infty$ we have $p = k_0$ for all values of k_0. $\Delta E_{\mathrm{ch}}(p)$ for $Q = 1$, $N = 6$ and four representative values of c are shown in figure 2(a). The dispersion is approximately parabolic.

Close to the Fermi level, ΔE_{ch} is proportional to $|p - p_{\mathrm{F}}|$. The proportionality constant is the Fermi velocity, which can be obtained via

$$v_{\mathrm{F}} = \left(\frac{\mathrm{d}\Delta E_{\mathrm{ch}}}{\mathrm{d}k_0}\right)_Q \Big/ \left(\frac{\mathrm{d}p}{\mathrm{d}k_0}\right)_Q \tag{4.4}$$

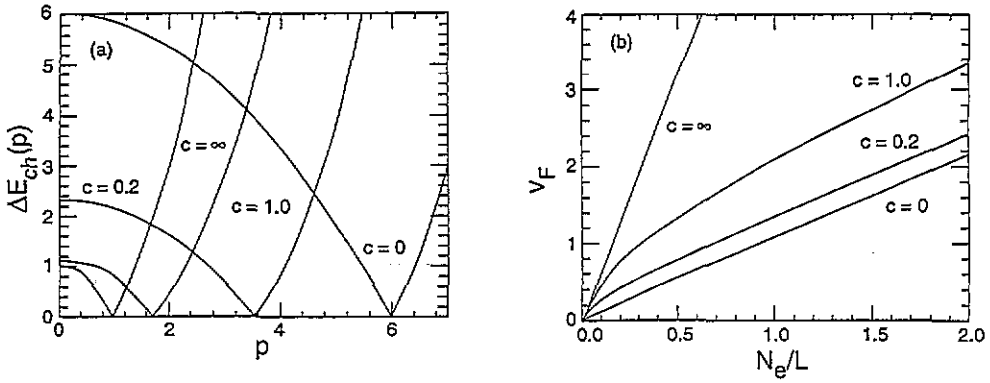

Figure 2. (a) Charge excitation spectrum for $N = 6$ and $Q = 1$ for four repulsive coupling strengths, $c = 0$, $c = 0.2$, $c = 1.0$, and $c = \infty$. ΔE_{ch} vanishes at the Fermi surface, characterized by $p_F = \pi N_e/L$. Here $|p| < p_F$ corresponds to hole states, while $|p| > p_F$ represents particle states. $c = 0$ corresponds to free charges, and $c = \infty$ to a hard-core potential among the electrons. (b) The Fermi velocity as a function of band filling for the four c values considered above. Note that as long as c is finite the curves have slope as for $c = 0$ if N_e/L is large and the same slope as the $c = \infty$ curve if Q is small.

where $(\mathrm{d}p/\mathrm{d}k_0)_Q = 2\pi\rho(Q)$. To obtain the numerator an additional differential equation for $(\mathrm{d}\epsilon/\mathrm{d}k_0)_{k_0=Q}$ has to be solved. In the limit $c \to \infty$ we obtain $v_F = 2Q = 2\pi N_e/L$, while for $c \to 0$ we have $v_F = 2Q = 2\pi N_e/LN$. In this latter case we have to distinguish v_F for $|k_0| > Q$ and $|k_0| < Q$, since $\rho(k) + \rho_h(k)$ is discontinuous at the Fermi level, but the Fermi velocity for particles and holes is the same. The Fermi velocity for $N = 6$ and four coupling strengths c is shown in figure 2(b) as a function of N_e/L. For a large band filling and finite c, the slope of v_F with N_e/L is the same as for the $c = 0$ situation, while if Q is small or comparable to c this slope is that of hard-core fermions.

The energy of the elemental spin-wave excitations is given by

$$\Delta E_S^{(l)}(\xi_0) = |\varphi^{(l)}(\xi_0)|. \tag{4.5}$$

The excitations correspond to a spin flip without changing the number of electrons. There are $N - 1$ spin-wave branches, one corresponding to each spin degree of freedom. The momentum of the spin wave is given by ($B_l = \infty$)

$$p_l(\xi_0) = 2\pi \int_{-\infty}^{\xi_0} \mathrm{d}\xi\ \sigma^{(l)}(\xi). \tag{4.6}$$

It is clear that $p_l \to 0$ as $\xi_0 \to -\infty$; on the other hand, for $\xi_0 \to \infty$ we obtain $p_l^{\max} = 2\pi(1 - l/N)N_e/L$. This constraints the range of the momentum of the spin waves, which in this way is correlated to the Fermi surface of the charges.

Let us now consider the limits $c \to 0$ and $c \to \infty$. For $c \to 0$ only excitations within the range $|\xi_0| \leqslant Q$ are meaningful, and we obtain

$$\Delta E_S^{(l)}(\xi_0) = (N - l)(Q^2 - \xi_0^2) \qquad p_l = (N - l)(\xi_0 + Q). \tag{4.7}$$

For a strict hard-core potential ($c \to \infty$), on the other hand, the excitation energy vanishes for all branches and all momenta. Hence, the spin-singlet state is not the ground-state of the system if $c \to \infty$; an infinitesimal field would align all spins into one direction.

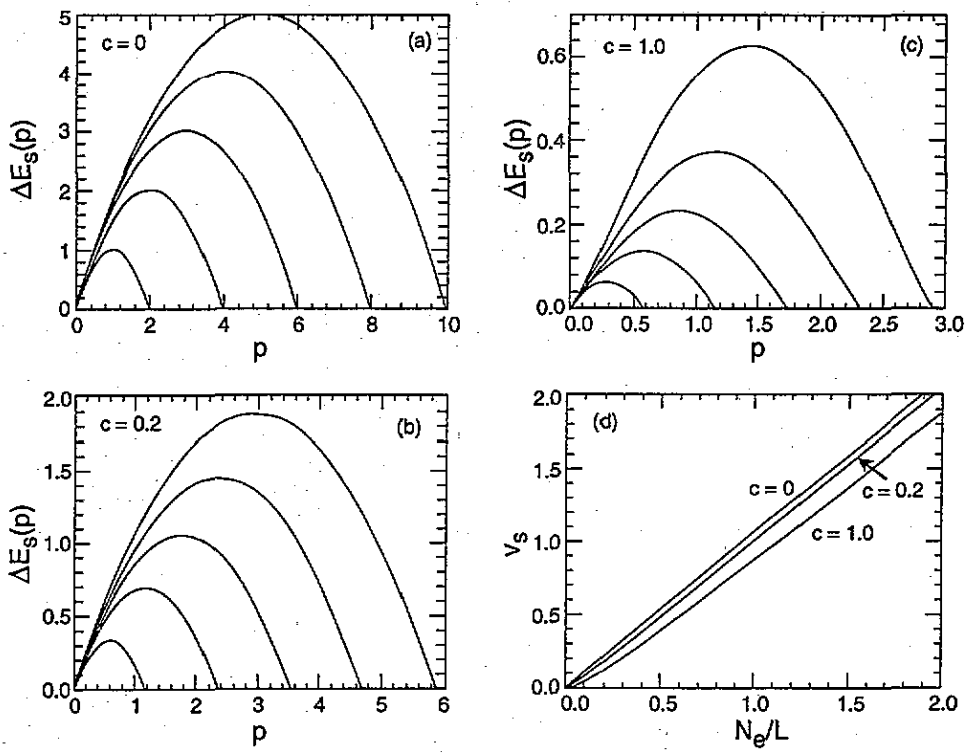

Figure 3. Spinwave excitations for $N = 6$ and $Q = 1$ for three repulsive coupling strengths: (a) $c = 0$, (b) $c = 0.2$ and (c) $c = 1$. There are $N - 1$ (i.e. five) spin-wave branches with a common spin-wave velocity. The maximum momentum of each branch is given by $p_l^{\max} = 2\pi(1 - l/N)N_e/L$. The dispersion can be characterized by three quantities: v_S, $\Delta E_S^{(l)}(\zeta = 0)$ and $p_l^{\max}$. (d) The spin-wave velocity as a function of electron density for the same values of the coupling. v_S grows monotonically with the electron density and decreases monotonically with increasing c. If c is finite the curves have the same slope as for $c = 0$ if N_e/L is large, while if $c > Q$ the spin-wave velocity softens. For $c \to \infty$ we have that $v_S = 0$ for all electron densities.

Our numerical results for the spin-wave dispersions are shown in figure 3(a)–(c) for $N = 6$ and $Q = 1$, and the coupling strengths $c = 0$, $c = 0.2$ and $c = 1$, respectively. Since $N = 6$ there are five branches for each case. Note that $\Delta E_S^{(l)}$ is an even function of ξ_0 and $p_l(\xi_0) - \pi(1 - l/N)(N_e/L)$ is odd, and

$$\frac{\Delta E_S^{(l)}(\xi_0 = 0)}{\Delta E_S^{(l')}(\xi_0 = 0)} = \frac{p_l^{\max}}{p_{l'}^{\max}} = \frac{N - l}{N - l'}. \tag{4.8}$$

The spin-wave velocity is obtained from the long-wavelength limit of the spectrum, i.e. as $\xi_0 \to -\infty$. All branches have the same v_S, which is inversely proportional to the magnetic susceptibility,

$$v_S \chi_S = \frac{N^2 - 1}{12\pi}. \tag{4.9}$$

The product $v_S \chi_S$ depends only on N and not on c nor on the band-filling.

In the limit $c \to 0$ $v_S = (2\pi/N)(N_e/L)$, while for $c \to \infty$ we have $v_S = 0$. The spin-wave velocity for $N = 6$ is displayed in figure 3(d) as a function of the electron density for three values of c. v_S grows monotonically with N_e/L and decreases monotonically with c if N_e/L is kept constant. Again, if $c \ll Q$ the slope of v_S with N_e/L is parallel to the $c = 0$ curve, while for $c > Q$ the spin-wave velocity gradually becomes soft.

5. Ground-state Bethe *ansatz* equations: attractive interaction

The discrete Bethe *ansatz* equations derived by Sutherland [4] diagonalize model (1.1) for repulsive and attractive interaction. For an attractive potential, $c < 0$, fermions of different colour tend to form bound states, so that the rapidities corresponding to the ground-state solution are in general complex [2,6]. For N different spin components we may have bound states of up to N fermions. In the thermodynamic limit a bound state of n electrons ($n \leqslant N = 2S + 1$) is characterized by one real rapidity $\zeta^{(n-1)}$ (which parametrizes the motion of the centre of mass of the bound state) and in general complex rapidities, $\xi^{(l)}$, given by

$$\begin{aligned} &\xi_p^{(l)} = \zeta^{(n-1)} + \mathrm{i}pc/2 \qquad l \leqslant n-1 \leqslant 2S \\ &p = -(n-l-1), -(n-l-3), \ldots, (n-l-1) \end{aligned} \tag{5.1}$$

where we denote the k rapidities by $\xi^{(0)}$. Since all rapidities within a given set have to be different to ensure linearly independent solutions, they satisfy Fermi statistics, i.e. a rapidity is either represented (particle) or missing (hole). The ground-state integral equations can then be formulated in terms of energy potentials (entering the Fermi distribution) for each class of states, $\epsilon^{(l)}(\zeta)$, where $l = 0, \ldots, n-1$. A negative energy potential defines the particles of the band and a positive potential corresponds to holes. For a pure Zeeman splitting we have [7]

$$\begin{aligned} \epsilon^{(l)}(\zeta) = (l+1)\left[\zeta^2 - \frac{l(l+2)}{12}c^2 - \mu\right] - \frac{1}{2}(l+1)(2S-l)H - \sum_{q=0}^{2S}\int_{-B_q}^{B_q} \mathrm{d}\zeta'\, \epsilon^{(q)}(\zeta') \\ \times \int \frac{\mathrm{d}\omega}{2\pi} \exp\bigl(\mathrm{i}(\zeta-\zeta')\omega - (l+q-p_{l,q})|\omega c|/2\bigr) \\ \times \sinh\bigl((p_{l,q}+1)\omega c/2\bigr)/\sinh(\omega c/2) \end{aligned} \tag{5.2}$$

where $p_{l,q} = \min(l,q) - \delta_{l,q}$ and μ is the chemical potential. The $\epsilon^{(l)}(\zeta)$ are monotonically increasing functions of $|\zeta|$ with zeros at $\pm B_l$, i.e. $\epsilon^{(l)}(\pm B_l) = 0$. These zeros correspond to the 'Fermi surfaces' of the various spin-charge bound states and determine the integration limits with respect to ζ' in (5.2), which are integrations over 'particle' states. If we invert the above integral equations, so that the integrals on the right-hand side are over 'hole' states, then the integration kernel is again of the form (2.2) as dictated by the SU(N) symmetry. In contrast to the repulsive case, all B_l are finite if the interaction is attractive (otherwise the density of fermions would diverge).

The distribution densities, $\rho^{(l)}(\zeta)$ for particles and $\rho_h^{(l)}(\zeta)$ for holes, for each class of rapidities can be obtained from the energy potentials by differentiating with respect to μ,

$$\rho_h^{(l)}(\zeta) + \rho^{(l)}(\zeta) = -(1/2\pi)\ (\partial\epsilon^{(l)}/\partial\mu) \tag{5.3}$$

for $l = 0, \ldots, N-1$. Differentiation of equations (5.2) with respect to μ yields the integral equations satisfied by the distribution densities

$$\rho_{\rm h}^{(l)}(\zeta) + \rho^{(l)}(\zeta) = \frac{(l+1)}{2\pi} - \sum_{q=0}^{2S} \int_{-B_q}^{B_q} {\rm d}\zeta' \, \rho^{(q)}(\zeta') \int \frac{{\rm d}\omega}{2\pi} \exp\bigl({\rm i}(\zeta - \zeta')\omega$$

$$- (l + q - p_{l,q})|\omega c|/2\bigr) \frac{\sinh\bigl((p_{l,q}+1)\omega c/2\bigr)}{\sinh(\omega c/2)}. \tag{5.4}$$

Equations (5.4) have been derived previously by Takahashi [6].

The ground-state energy is given by [6] (L is the length of the box)

$$E/L = \sum_{l=0}^{2S} (l+1) \int_{-B_l}^{B_l} {\rm d}\zeta \left[\zeta^2 - \frac{l(l+1)}{12} c^2\right] \rho^{(l)}(\zeta) \tag{5.5a}$$

the number of particles of each spin component is determined by

$$n_l = \sum_{q=2S-l}^{2S} \int_{-B_q}^{B_q} {\rm d}\zeta \; \rho^{(q)}(\zeta) \tag{5.5b}$$

and the total number of electrons (or in general fermions) becomes

$$N_{\rm e}/L = \sum_{l=0}^{2S} n_l = \sum_{l=0}^{2S} (l+1) \int_{-B_l}^{B_l} {\rm d}\zeta \rho^{(l)}(\zeta). \tag{5.5c}$$

6. Ground state properties: attractive interaction

In zero field all spin projections have the same number of particles, so that from (5.5b) $B_l = 0$ for all $l < 2S$. The only non-vanishing particle density is then $\rho^{(2S)}(\zeta)$, which corresponds to bound states of N fermions all of different colour (e.g. for electrons with $N = 2$ the bound states can be interpreted as Cooper pairs, while for nucleons with $N = 4$ they represent α-particles). We denote B_{2S} by Q. Equations (5.2) and (5.4) reduce to single integral equations of the Fredholm type with the kernel given by

$$K_S(\zeta) = \frac{1}{\pi|c|} {\rm Re}\Bigl[\Psi(N + {\rm i}\zeta/|c|) - \Psi(1 + {\rm i}\zeta/|c|)\Bigr] \tag{6.1}$$

where Ψ denotes the digamma function and Re stands for real part. Note that if ζ is scaled with $|c|$ the integral equation for $\rho^{(2S)}$ for a given N depends only on the parameter $Q/|c|$, which parametrizes the total number of fermions.

In the absence of a field the integral equation for the density $\rho^{(2S)}$ can be solved analytically in the limit $c \to 0$: $\rho^{(2S)}(\zeta) = 1/(2\pi)$ for $|\zeta| \leqslant Q$ and $\rho_{\rm h}^{(2S)}(\zeta) = N/(2\pi)$ if $|\zeta| \geqslant Q$, and zero elsewhere. The corresponding energy potential is $\epsilon^{(2S)}(\zeta) = (\zeta^2 - Q^2)$ if $|\zeta| \leqslant Q$ and $\epsilon^{(2S)}(\zeta) = N\,(\zeta^2 - Q^2)$ if $|\zeta| \geqslant Q$. It follows that

$$N_{\rm e}/L = \frac{N}{\pi} Q \qquad E/L = \frac{N}{3\pi} Q^3 \qquad \mu = Q^2. \tag{6.2}$$

This result corresponds to a free fermion gas with N colour components. For $c \to -\infty$, on the other hand, we obtain $\rho^{(2S)}(\zeta) = N/(2\pi)$ for $|\zeta| \leqslant Q$ and $\rho_h^{(2S)}(\zeta) = N/(2\pi)$ if $|\zeta| \geqslant Q$, and zero elsewhere. From (5.5*a*) it follows that the ground-state energy is not bound from below as $c \to -\infty$, so that we have to rescale energies with c^2:

$$\frac{N_e}{L} = \frac{N^2}{\pi} Q \qquad \frac{E}{Lc^2} = \frac{N^2}{3\pi}\left[\left(\frac{Q}{c}\right)^2 - S(S+1)\right]Q \qquad \frac{\mu}{c^2} = \left(\frac{Q}{c}\right)^2 - \frac{S(S+1)}{3}. \tag{6.3}$$

Hence, the spin-neutral bound states are very stable if the attractive potential is very strong.

For general c the integral equation for $\epsilon^{(2S)}(\zeta)$ has to be solved numerically (discretizing the integral) for a given Q and μ is determined by the zero of the potential. In figure 4(a) we present the density function $\rho^{(2S)}(\zeta) + \rho_h^{(2S)}(\zeta)$ as a function of ζ for $Q = 1$, $N = 4$ and four different values of c. For $c \neq 0$ the density function smoothly interpolates between particle and hole states, but for $c = 0$ the distribution is discontinuous at the Fermi level. The particle density, N_e/L, is displayed as a function of Q in figure 4(b) for $N = 4$ and the same values of c. The curves for finite c have the same slope as the $c = 0$ line if $Q \gg |c|$, but if $Q < |c|$ the slope approaches the one of the $c = -\infty$ line. In figure 4(c) we show the energy density as a function of N_e/L. For $c = 0$ it simply corresponds to a cubic parabola with a minimum of $E = 0$ at $N_e/L = 0$. For $N_e/L \neq 0$ the minimum is at a finite density of electrons. The binding energy of the bound states increases with $|c|$. The chemical potential μ is determined from $\epsilon^{(2S)}(Q) = 0$, i.e.

$$\mu = Q^2 - \frac{S(S+1)}{3}c^2 - \frac{1}{N}\int_{-Q}^{Q} d\zeta\ \epsilon^{(2S)}(\zeta)\ K_S(Q-\zeta) \tag{6.4}$$

which for $c \to 0$ and $c \to -\infty$ reduces to (6.2) and (6.3), respectively. The chemical potential as a function of N_e/L for $N = 4$ is shown in figure 4(d). Due to the finite binding energy if $c \neq 0$ μ is negative for small energy densities.

The energy potentials of the spin-dependent bound states, $\epsilon^{(l)}(\zeta)$ for $l = 0, \ldots, 2S-1$, are determined from (5.2) as integrals over $\epsilon^{(2S)}(\zeta)$ (in the absence of external fields). They are all positive symmetric functions of ζ and are monotonically increasing with increasing $|\zeta|$. Hence, they have their minimum at $\zeta = 0$. In the limit $c \to 0$ we obtain for the potentials

$$\epsilon^{(l)}(\zeta) = \begin{cases} 0 & \text{if} \quad |\zeta| \leqslant Q \\ (l+1)(\zeta^2 - Q^2) & \text{if} \quad |\zeta| \geqslant Q. \end{cases} \tag{6.5}$$

For very large negative c the energy potentials have a very large positive energy.

In zero field the spin-dependent bound states always have a positive potential, so these bands are not populated in the ground-state. This is consistent with the original assumption that in the ground-state there are only spin-neutral bound states of N fermions. It requires a finite excitation energy to overcome the energy of the spin gaps. This energy could be provided by external fields; for a pure Zeeman field the critical magnetic fields required to break the spin-neutral charge bound state are given by the minimum of the potentials at $\zeta = 0$,

$$H_c^{(l)} = -\frac{2}{2S-l}\left(\mu + \frac{l(l+2)}{12}c^2\right) - \frac{2}{(l+1)(2S-l)}\int_{-Q}^{Q} d\zeta\ \epsilon^{(2S)}(\zeta)\frac{1}{\pi|c|} \times \mathrm{Re}\left[\Psi\left(\frac{N+l+1}{2} + i\frac{\zeta}{|c|}\right) - \Psi\left(\frac{N-l-1}{2} + i\frac{\zeta}{|c|}\right)\right]. \tag{6.6}$$

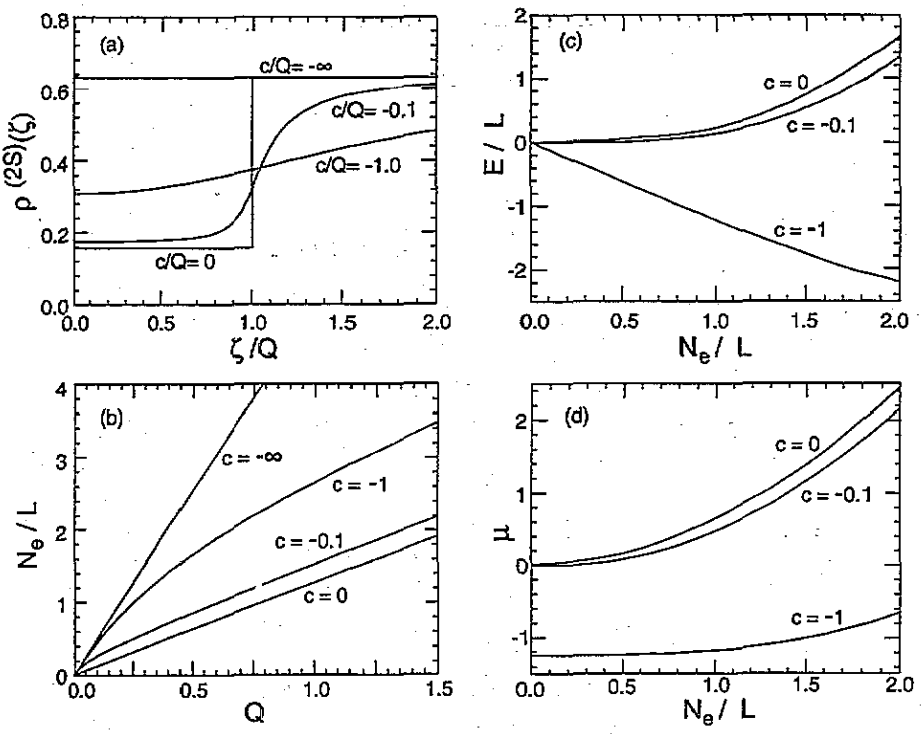

Figure 4. (a) Density distribution function of the rapidities corresponding to the spin- neutral bound states of N electrons for an attractive δ-function potential. Q is the integration limit determining the Fermi surface. The curves are for $N = 4$ and four interaction strengths c. A step function at $\zeta = \pm Q$ is obtained for non-interacting fermions. (b) Electron density N_e/L as a function of Q for $N = 4$ and four different couplings. Note that for finite c the curves have the same slope as for $c = 0$ if Q is large and the slope of the $c = -\infty$ line if Q is small. (c) Energy density and (d) chemical potential for $N = 4$ and $Q = 1$ in zero field as a function of the particle density for three interaction strengths. For $c = -\infty$ the energy and the chemical potential are not bound from below. Both the energy and the chemical potential can be negative as a consequence of the finite binding energy for the bound states.

All critical fields vanish as $c \to 0$. In the limit $Q = 0$ we obtain the following analytic expression:

$$H_c^{(l)} = \frac{N^2 - 1 - l(l+2)}{6(2S - l)} c^2. \tag{6.7}$$

The critical fields as a function of c are shown in figure 5 for $Q = 1$ (solid curves) and $Q = 0$ (dashed parabolas). Hence, for given c and Q the system does not respond to a field smaller than the lowest critical field, i.e. $H_c^{(0)}$. For a field slightly larger than this critical one the $l = 0$ energy band begins to be populated, i.e. some of the spin-neutral bound states of N electrons are broken up into N unpaired propagating electron states, all with the maximum spin-component S. Hence, for $H > H_c^{(0)}$ the system has a finite magnetization, which for a field slightly larger than $H_c^{(0)}$ is proportional to $(H - H_c^{(0)})^{1/2}$, as a consequence of the van Hove singularity of the empty $l = 0$ band.

For $S = \frac{1}{2}$ the lack of response to a magnetic field smaller than a critical one is in part reminiscent of the Meissner effect, except for the lack of diamagnetism, which is not

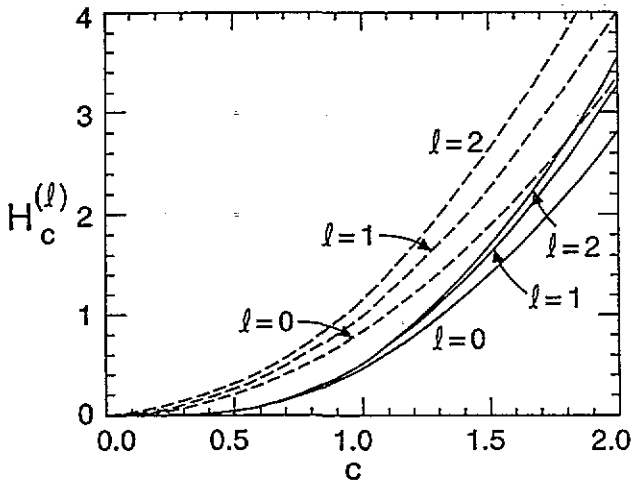

Figure 5. Critical fields $H_c^{(l)}$, $l < 2S$, as a function of interaction strength $|c|$ for $N = 4$. The solid curves correspond to $Q = 1$, while the dashed ones to $Q = 0$ (the very-low-electron-density limit). Note that $H_c^{(0)}$ is the lowest field and hence the spin-neutral bound states break up into N independent propagating states if $H > H_c^{(0)}$.

defined in one dimension. The 'Cooper pairs' are then gradually broken up by a field larger than the critical one. Note that there is no long-range order in one dimension, the bound states do not cease to exist at finite T (the critical temperature is zero), and in contrast to the BCS theory there is no condensation of Cooper pairs. For $N = 4$ the four degrees of freedom could arise from the direct product of a spin $\frac{1}{2}$ and an isospin $\frac{1}{2}$; the spin-neutral bound states then represent α-particles in a sea of nucleons.

If the population of the $l = 0$ band (unpaired electrons) is finite the integral equations (5.2) no longer reduce to a single integral equation of the Fredholm type, but to two coupled ones. The finite band-filling, i.e. $B_0 \neq 0$, slightly renormalizes the critical field for the second band of spin-dependent states to be populated, which corresponds to $l = 1$, i.e. $H_c^{(1)}$ is slightly changed. A small occupation of this band also contributes to the magnetization with a term proportional to $(H - H_c^{(1)})^{1/2}$, due to the one-dimensional van Hove singularity of the empty band, as do all the other bands with $l < 2S$. In summary, the $T = 0$ susceptibility vanishes for $H < H_c^{(0)}$ and has a square-root singularity each time the field equals a critical one.

The low-temperature specific heat is proportional to T, as a consequence of the Fermi–Dirac statistics obeyed by the rapidities. In zero field the only Fermi surface contributing to the specific heat is that of the spin-neutral bound states ('Cooper pairs' for $S = \frac{1}{2}$, α-particles for $N = 4$; note that the Fermi statistics prevents the Bose condensation). In the presence of an external magnetic field the specific heat is still proportional to T, unless the field is equal to a critical one, where the van Hove singularity of the empty band gives rise to a $T^{1/2}$ term.

7. Elemental excitations: attractive interaction

Pure charge excitations are obtained by adding or removing a spin-neutral boundstate of N electrons of different colours. This corresponds to adding or removing a string from

the set of strings of length $2S$. All other energy bands or sets of strings involve mixed spin and charge states, and for the sake of clarity we will call them spin excitations. We will restrict ourselves to excitations in the absence of field. Adding or removing a rapidity ζ from a given set gives rise to an additional driving term in equations (5.4), which is of the order of $1/L$ smaller than the main driving term. This additional driving term induces a change in the distribution density functions, $\rho^{(l)}(\zeta)$, and hence a change in the energy, which is the excitation energy. Since the integral equations (5.4) are linear in the densities, the superposition principle applies, i.e. the excitation energies and their momenta are additive and independent (soliton-like behaviour). This is the traditional but tedious way of computing (i.e. solving the integral equations for the additional driving term) the excitation energies. The energies of the elemental excitations are also given by the energy potentials $\epsilon^{(l)}$, which are the energies entering the Fermi distribution function. The two approaches yield identical results.

In the absence of fields we have that $B_l = 0$ for $l = 0, \ldots, 2S-1$ and only Q is different from zero. If ζ_0 is the 'charge' rapidity under consideration, we have a 'particle' excitation if $|\zeta_0| > Q$ (a rapidity is added) and a hole excitation if $|\zeta_0| < Q$ (a rapidity is removed), i.e.

$$\Delta E_{\mathrm{ch}}(\zeta_0) = |\epsilon^{(2S)}(\zeta_0)| \tag{7.1}$$

where $\zeta_0 = \pm Q$ represents the Fermi surface. The numerical solution of the integral equation for $\epsilon^{(2S)}$ has been discussed in section 6. In the limit $c \to 0$ we obtain

$$\Delta E_{\mathrm{ch}}(\zeta_0) = \begin{cases} (Q^2 - \zeta_0^2) & \text{if} \quad |\zeta_0| < Q \\ N(\zeta_0^2 - Q^2) & \text{if} \quad |\zeta_0| > Q \end{cases} \tag{7.2}$$

in the limit $c \to -\infty$, on the other hand, we have $\Delta E_{\mathrm{ch}}(\zeta_0) = N|\zeta_0^2 - Q^2|$ for all ζ_0.

As already discussed in section 4 ζ_0 is not the physical momentum of the excitation. The momentum is determined from the density distribution functions via

$$p_{\mathrm{ch}}(\zeta_0) = 2\pi \int_0^{\zeta_0} \mathrm{d}\zeta \, \left[\rho^{(2S)}(\zeta) + \rho_{\mathrm{h}}^{(2S)}(\zeta)\right]. \tag{7.3}$$

Hence, $p_{\mathrm{ch}}(\zeta_0 = 0) = 0$, and the momentum is a monotonically increasing function of ζ_0 with the symmetry $p_{\mathrm{ch}}(-\zeta_0) = -p_{\mathrm{ch}}(\zeta_0)$. For $\zeta_0 = Q$ we obtain from (7.3) that the Fermi momentum is $p_{\mathrm{F}} = (\pi/N)(N_{\mathrm{e}}/L)$. In the limit $c \to 0$ we have that $p_{\mathrm{ch}} = \zeta_0$ for $|\zeta_0| < Q$ and $p_{\mathrm{ch}} = p_{\mathrm{F}} + N(\zeta_0 - Q)$ if $\zeta_0 > Q$. The charge excitation spectrum is approximately parabolic and is shown in figure 6(a) for $N = 4$, $Q = 1$ and four values of c.

The charge excitations vanish at the Fermi level. Close to the Fermi level the excitation energy is proportional to $(p - p_{\mathrm{F}})$, the proportionality constant being the Fermi velocity

$$v_{\mathrm{F}} = \left(\frac{\mathrm{d}\Delta E_{\mathrm{ch}}}{\mathrm{d}\zeta_0}\right)_Q \Big/ \left(\frac{\mathrm{d}p_{\mathrm{ch}}}{\mathrm{d}\zeta_0}\right)_Q . \tag{7.4}$$

From (7.3) the denominator is $2\pi\rho^{(2S)}(Q)$. In order to obtain the numerator an integral equation for $(\partial\epsilon^{(2S)}(\zeta_0)/\partial\zeta_0)|_{\zeta_0=Q}$ has to be solved numerically. An analytical solution can be obtained in the limits $c \to 0$, $v_{\mathrm{F}} = (2\pi/N)(N_{\mathrm{e}}/L) = 2p_{\mathrm{F}}$, and $c \to -\infty$, $v_{\mathrm{F}} = (2\pi/N^2)(N_{\mathrm{e}}/L) = 2p_{\mathrm{F}}/N$. For $c \to 0$ we have to distinguish the Fermi velocity for charge particle and hole excitations, since the density function is discontinuous in this

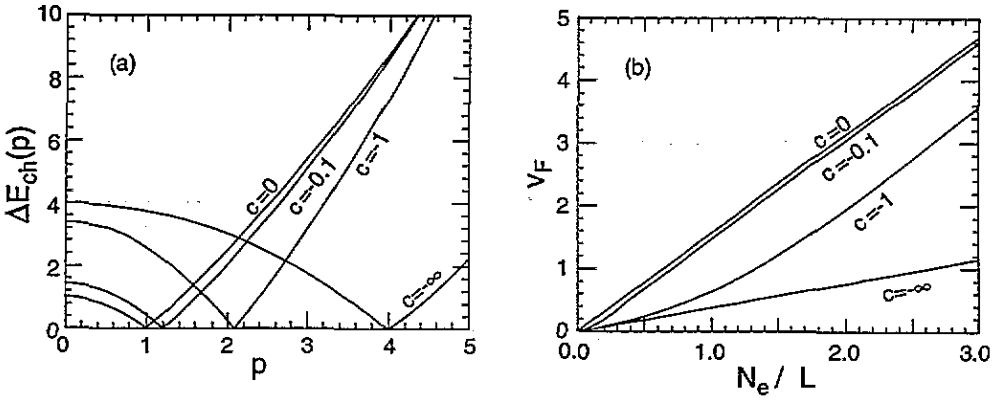

Figure 6. (a) Charge excitation spectrum for $N = 4$ and $Q = 1$ for four different values of the attractive interaction, $c = 0$, $c = -0.1$, $c = -1.0$ and $c = -\infty$. ΔE_{ch} vanishes at the Fermi surface, characterized by $p_F = (\pi/N)(N_e/L)$. The charge excitations correspond to removing (for a hole) one spin-neutral boundstate ($|p| < p_F$) or to adding one boundstate (for a particle) to the system ($|p| > p_F$). The dispersion is approximately parabolic. (b) Fermi velocity as a function of band filling for the four c values considered above. Note that if $|c| \gg Q$ the slope of the curves is approximately that corresponding to $c = -\infty$, while if $|c|$ is small compared to Q, the slope asymptotically approaches that of the $c = 0$ line.

limit, but of course the two cases yield the same v_F. The Fermi velocity for $N = 4$ and four interaction strengths is displayed in figure 6(b) as a function of band filling, N_e/L. If $|c| \ll Q$ the slope of v_F with N_e/L is the same as for $c = 0$, while if $|c| \gg Q$ (small electron densities) the slope approaches that of infinite interaction.

The elemental excitations corresponding to the remaining classes of rapidities, $l = 0, \ldots, 2S-1$, refer to spin-dependent bound states involving less than N electrons. Adding one rapidity, however, involves an actual addition of particles, so these states also carry a charge. As discussed above in the absence of an external field the bands $\epsilon^{(l)}$ for $l < 2S$ are completely empty in the ground-state and it requires a finite excitation energy to overcome the gap and populate them. Hence, excitations are obtained by adding one rapidity $\zeta_0^{(l)}$ and the excitation energies are given by

$$\Delta E_S^{(l)}(\zeta_0^{(l)}) = \epsilon^{(l)}(\zeta_0^{(l)}) \qquad l = 0, \ldots, 2S-1. \tag{7.5}$$

The physical momentum of the excitation is from its definition

$$p^{(l)}(\zeta_0^{(l)}) = 2\pi \int_0^{\zeta_0^{(l)}} \mathrm{d}\zeta \; \rho_h^{(l)}(\zeta) \tag{7.6a}$$

where the hole density function is given by (5.4). Hence, the momentum of the added boundstate is zero if incorporated into the centre of the band. Straightforward integration yields

$$p^{(l)}(\zeta_0^{(l)}) = (l+1)\zeta_0^{(l)} - 2\int_{-Q}^{Q} \mathrm{d}\zeta \rho^{(2S)}(\zeta) \times \mathrm{Im}\ \ln\left\{\Gamma\left[\frac{N+l+1}{2} + \mathrm{i}\frac{\zeta_0^{(l)}-\zeta}{|c|}\right] \Big/ \Gamma\left[\frac{N-l-1}{2} + \mathrm{i}\frac{\zeta_0^{(l)}-\zeta}{|c|}\right]\right\} \tag{7.6b}$$

where we used that $\rho^{(2S)}(\zeta)$ is an even function of ζ. In the limit $\zeta_0^{(l)} \to \infty$ expression (7.6*b*) reduces to

$$p^{(l)}(\zeta_0^{(l)}) = (l+1)(\zeta_0^{(l)} - p_F) \tag{7.6c}$$

where p_F is the Fermi momentum of the charge excitations. Hence for very large $\zeta_0^{(l)}$ the momentum is proportional to $\zeta_0^{(l)}$ and the spectrum is approximately parabolic.

In the limit $c \to 0$ both the excitation energy and the momentum vanish identically for $|\zeta_0^{(l)}| \leqslant Q$; expressing the excitation energy as a function of the momentum for $|\zeta_0^{(l)}| \geqslant Q$ we obtain

$$\Delta E_S^{(l)}(p) = 2Q|p| + p^2/(l+1) \qquad l = 0, \ldots, 2S-1. \tag{7.7}$$

For $c \to -\infty$, on the other hand, the spin-dependent excitations require an infinite energy, as a consequence of the very large chemical potential. This means that it is energetically unfavourable to have electrons in bound states involving less than N particles.

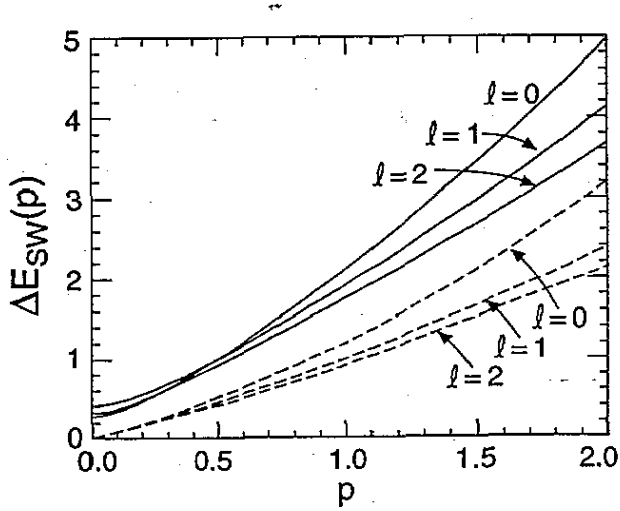

Figure 7. Excitation spectrum of spin-charge bound states for $N = 4$, $Q = 1$ and $c = -1$ (solid curves) and $c = 0$ (dashed curves). There are three $(N-1)$ branches of excitations. For $c < 0$ there is always a gap in the spectrum, since a finite energy is required to unbind the spin-neutral bound states of N fermions. On the other hand, if $c = 0$ there are long-wavelength excitations with zero energy and a spin-wave velocity ($v_S = 2Q$) can be defined.

The dispersion of the spin-charge bound-state excitations is displayed in figure (7) for $N = 4$, $Q = 1$ and two values of the interaction, $c = 0$ and $c = -1$. For $c < 0$ all branches require a finite excitation energy for their population; the gaps are given by the minimum at $p = 0$ and correspond to the critical fields discussed in section 6 (see figure 5). The dispersion is approximately parabolic and crossovers of bands (without hybridization) may occur. A spin-wave velocity can only be defined for $c = 0$ (for $c \neq 0$ there is no Fermi surface) at the point $p = 0$, $v_{sw} = 2Q$. This result agrees with the $c \to 0$ limit for the repulsive interaction discussed in section 4.

8. Concluding remarks

We have studied some ground-state properties and the spectrum of elemental charge and spin excitations of the N-component Fermi gas interacting via a δ-function potential in one dimension. The model is integrable and has been diagonalized by means of a sequence of N nested Bethe *ansatze* by Sutherland [4] (see also [5]). For a *repulsive* interaction the electrons do not form bound states in the ground-state and the system consists of freely propagating charge states (characterized by the charge rapidities) and spin-waves (described by real spin rapidities). As usual for one-dimensional systems charge and spin degrees of freedom decouple and propagate with different velocities. For an *attractive* interaction potential, on the other hand, the ground-state in the absence of symmetry breaking fields consists of spin-neutral charge bound states of N particles, each fermion with a different spin-component. For $N = 2$ these bound states can be interpreted as Cooper pairs, and for $N = 4$ as α-particles in a sea of nucleons with the four internal degrees of freedom arising from the direct product of the spin $\frac{1}{2}$ and an isospin $\frac{1}{2}$.

For a *repulsive* interaction we calculated, by numerically solving the ground-state Bethe *ansatz* integral equations, the ground-state energy, the chemical potential and the magnetic susceptibility as a function of band filling for $N = 6$ and several coupling strengths. The one-dimensional van Hove singularity determines the properties (i.e. the zero-field susceptibility and the γ coefficient of the specific heat are both proportional to Q^{-1}) when the band is nearly empty. A small (but arbitrary) symmetry-breaking field induces logarithmic field singularities in the susceptibility [21], and the γ-coefficient of the specific heat is singular in the sense that the field and temperature tending to zero limits cannot be interchanged [25]. In the limit $N \to \infty$ the fermion character of the particles (Pauli's principle) is not relevant (particles of different colours are distinguishable) and the integral equation for the charges reduces to that of a gas of bosons [1, 4].

We also calculated the spectrum of elemental excitations in the absence of external fields. There are two ways to obtain the excitation energies, (i) through their energy potentials and (ii) by solving the integral equation for the change in the distribution densities when a rapidity is either added or removed from the system. The two methods yield identical results. The charge excitations are approximately parabolic in the momentum and have a Fermi surface with the Fermi momentum related to the band filling. The Fermi velocity is roughly proportional to N_e/L. There are $N - 1$ spin-wave excitation branches. The excitation energy of long-wavelength spin waves is proportional to the momentum. This defines the spin-wave velocity, which is the same for all branches. The product of the spin-wave velocity and the zero-field susceptibility yields a constant which is independent of the coupling strength and of the band-filling. The momentum range of the spin-wave spectrum is correlated with the Fermi surface of the charges. For $c \to \infty$ the spin waves become soft ($v_S \to 0$) and the susceptibility and the γ coefficient of the specific heat diverge for all band fillings.

For an *attractive* interaction the ground state, in the absence of external fields, consists of bound states of N particles. Although these charge bound states are a consequence of a coherent collective state, there is no long-range order in the system. The charge bound states have a finite binding energy, and a finite external symmetry-breaking potential (for instance a magnetic field in the case of Cooper pairs, reminiscent of the Meissner effect) is required to break the bound state into smaller units (e.g. an α-particle into nucleons or deuterons, or a Cooper pair into two propagating electrons). The charge excitations have a dispersion that is approximately parabolic, they have a Fermi surface with Fermi momentum and Fermi velocity determined by the band-filling. Rather than spin waves the system has

spin-charge excitations. These correspond to adding (or removing) a bound state with less than N particles. Since it requires a critical field to break the charge bound states, the excitation spectrum of the spin-charge bound states has an energy gap, but is otherwise roughly parabolic in the momentum. The $T = 0$ magnetic susceptibility is zero and a spin-wave velocity cannot be defined (except for $c = 0$). The specific heat is again linear in T, with the γ-coefficient determined by the Fermi velocity of the charge excitation spectrum.

In summary, if $c > 0$ the system behaves like a Luttinger liquid, while for $c < 0$ there are spin gaps in the excitation spectrum. This behaviour is qualitatively analogous to that of the $t - J$ model with low electron density around the point of supersymmetric coupling, $t = J$. Here a Luttinger liquid is expected when $t > J$, while spin gaps occur for $t < J$ [10].

Acknowledgment

The support of the US Department of Energy under grant DE-FG05-91ER45443 is acknowledged.

References

[1] Lieb E H and Liniger W 1963 *Phys. Rev.* **130** 1605
[2] Gaudin M 1967 *Phys. Lett.* A **24** 55
[3] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
[4] Sutherland B 1968 *Phys. Rev. Lett.* **20** 98
[5] Gu C H and Yang C N 1989 *Commun. Math. Phys.* **122** 105
[6] Takahashi M 1971 *Prog. Theor. Phys.* **46** 1388
[7] Schlottmann P 1993 *J. Phys.: Condens. Matter* **5** 5869
[8] Schlottmann P 1987 *Phys. Rev.* **36** 5177
[9] Lee K and Schlottmann P 1988 *J. Physique Coll.* **49** C8 709
Schlottmann P 1992 *J. Phys.: Condens. Matter* **4** 7565; 1993 *J. Phys.: Condens. Matter* **5** 313
[10] Schlottmann P 1990 *Phys. Rev.* B **41** 4164
[11] Sutherland B 1975 *Phys. Rev.* B **12** 3795
[12] Tsvelik A M and Wiegmann P B 1983 *Adv. Phys.* **32** 453
Andrei N, Furuya K and Lowenstein J H 1983 *Rev. Mod. Phys.* **55** 331
[13] Schlottmann P 1983 *Phys. Rev. Lett.* **50** 1697; 1989 *Phys. Rep.* **181** 1
[14] Choy T C and Haldane F D M 1982 *Phys. Lett.* **90A** 83
[15] Lee K and Schlottmann P 1989 *Phys. Rev. Lett.* **63** 2299
Schlottmann P 1991 *Phys. Rev.* **43** 3101
[16] Takahashi M 1970 *Prog. Theor. Phys.* **44** 348
[17] Schlottmann P 1992 *Phys. Rev.* B **45** 5784
[18] Sutherland B 1978 *Rocky Mountain J. Math.* **8** 413
[19] Calogero P, Ragnisco O and Marchioro C 1974 *Lett. Nuovo Cimento* **13** 383
[20] Mertens F G 1984 *Z. Phys.* B **55** 353
[21] Schlottmann P 1991 *Physical Phenomena at High Magnetic Fields* ed E Manousakis *et al* (Redwood City: Addison-Wesley) p 502
[22] Schlottmann P 1992 *Phys. Rev.* B **45** 5293
[23] Babujian H M 1983 *Nucl. Phys.* B **215** 317
[24] Andrei N and Lowenstein J H 1979 *Phys. Rev. Lett.* **43** 1698; 1980 *Phys. Lett.* B **90** 106
[25] Lee K and Schlottmann P 1989 *J. Phys.: Condens. Matter* **1** 2759
[26] des Cloizeaux J and Pearson J 1962 *Phys. Rev.* **128** 2131
[27] Johannesson H 1986 *Nucl. Phys.* B **270** 235
[28] Coll C F III 1974 *Phys. Rev.* B **9** 2150

[29] Woynarovich F 1982 *J. Phys. C: Solid State Phys.* **15** 85; 1982 *J. Phys. C: Solid State Phys.* **15** 97; 1983 *J. Phys. C: Solid State Phys.* **16** 5293
[30] Lee K and Schlottmann P 1989 *J. Phys.: Condens. Matter* **1** 10 193
[31] Bares P A and Blatter G 1990 *Phys. Rev. Lett.* **64** 2567
Bares P A, Blatter G and Ogata M 1991 *Phys. Rev.* B **44** 130